On Inconsistencies of Risk Adjusted Returns with Expected Utility Models in Optimization

A Skepticism about Optimizing RARs
in ERM Frameworks

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1. Terminology of Risk Adjusted Return (RAR)

- RAR in a narrow sense:
  - Return after deduction of expected loss (EL)
  - Numerator in RAROC
- RAR in the optimization context:
  - An objective mean-risk utility model denoted by $R(X):= r(\rho(X),\mathbb{E}[X])$ for r.v. X
  - RARs are represented by following measures:
    - Return on risk capital (RORC):
      - A ratio of mean ($\mathbb{E}[X]$) over risk capital ($\rho(X)$)

        $$R(X):= \mathbb{E}[X] / \rho(X)$$
      - A generic form for division-type RAR including RORAC and RAROC
    - Economic income created (EIC):
      - A mean ($\mathbb{E}[X]$) after deduction of cost of risk capital ($c\cdot\rho(X)$)

        $$R(X):= \mathbb{E}[X] - c\cdot\rho(X)$$
      - A generic form for subtraction-type RAR including EVA

2. Backgrounds

- Lam (2003) explains RARs as metrics for risk optimization analytics.
- RARs are vital in some influential ERM frameworks for insurers, for instance,
  - A.M. Best (2013) evaluates business strategy and capital allocation based on RARs as a strong characteristic of insurers.
  - S&P(2013) classifies optimizing RARs of insurers as a vital requirement for the top two ERM-rating categories since the first version of the criteria in 2005.
- What is the reason for above recommendations?

3. Motivation

- Possible reasons for the recommendations on optimizing RARs
  - Similarity to optimizing ROE;
  - Consistency with diffused capital allocation principle in ERM
- Optimizing ROE is not necessarily recommendable due to its shortcomings:
  - ROE encourages short-termism.
  - ROE discourages solvency.
  - ROE can be manipulated with debt-equity ratio.
- Tasche (1999) demonstrates that Euler-capital-allocation-principle is the only principle consistent with maximizing RORC, while maximizing RORC itself has not been validated.
- ICP16 (IAIS, 2015) recommends RARs only for remuneration use.

The validity of optimizing RARs in the ERM practice needs to be scrutinized!

4. Basic Assumption of the Investment Universe

- **Assumption:** To satisfy comparability of RAR,

  every portfolio X in the investment universe S satisfies following properties:

  - X is represented as a bounded random variable on $(\Omega, \mathcal{F}, P)$, corresponding to its monetary return.
  - X has the same principal amount with an acceptance of allocation in cash to cope with different granularity of investment items.
  - X is in one-to-one correspondence with a point on the relevant risk-mean plane, where cash corresponds to (0,0).

5. Search for Validation Benchmark

- The validation bench mark must reflect following properties:
  - RARs are cardinal measures using objective probability.
  - Optimization must be based on a normative, non-descriptive, model.
- We adopt vNM expected utility (EU) model as the validation benchmark only for the reason that it is a normative model and a cardinal measure using objective probability.
- However, specific utility function is not to be assumed since RARs are inherently needless for the investors who know their actual utility functions.
- Utility functions are only assumed to be continuous and increasing for a generosity in validation.
- Then, the validation of optimizing RAR will be given by the existence of EU which has consistency with the RAR.

6. Shortcomings of Previous EU Preference Studies

- Many papers have discussed stochastic models consistent with EU preference using weak inequalities (≥).
- EU preferences have consistencies with stochastic dominances
  - X dominates Y by FSD *iff* $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all non-decreasing utility $u(\cdot)$.
  - X dominates Y by SSD *iff* $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all concave utility $u(\cdot)$.
- EU preference with weak inequalities has following shortcomings:
  - Complete ordering by stochastic dominance means placing an impractical restriction on the investment universe.
  - Weak inequality lacks exactness which is required for the optimization.
- Above concerns are partly shared by Fishburn (1977).

7. Fishburn (1977)

- Fishburn(1977) defines a strict consistency with strict inequality called "congruence" and discusses the consistency without impractical restriction on the investment universe.
- Fishburn (1977) demonstrates that
  - The EU model can be expressed by EIC if the EU model has congruent mean-risk utility model using lower-partial-moments (k-th LPM: $\int_{X<m} (m-X)^k dP$) for all bounded random variables.
  - Corresponding utility functions are kinked (i.e., not differentiable but rather continuous and increasing).

8. Definitions of the Validation

- **Definition 1 (Fishburn):** A RAR denoted by R(·) is said to be congruent with an EU model and vice-versa, for all bounded random variables X and Y, if we have the equivalence:

$$R(X) > R(Y) \textit{ iff } \mathbb{E}[u(X)] > \mathbb{E}[u(Y)].$$

- **Definition 2:** Optimizing RAR is said to be valid if the RAR has any congruent EU model using continuous increasing utility.

9. Modification of Fishburn's Scope of Risk Measure

- Fishburn(1977) is limited in scope by mean-risk utility models with LPMs that are not monetary risk measures except for 1st LPM.
- However, dominant risk measures used for RARs in the ERM practice are monetary risk measures used to obtain risk capital, represented by VaR and t-VaR.
- We focus on the positive homogeneity among the axioms of coherent risk measure.
- The positive homogeneous class of risk-measures includes VaR, t-VaR, and 1st lower-partial-raw-moment (1st LPRM, i.e., 1st LPM around zero).

10. Introduction of Scale Properties

- Scale properties of RARs with positive homogeneous risk measures correspond to changes in cash position which is allowed by the Assumption.
- We denote by $\alpha X \in S$, $\alpha \in (0,1]$, investment in portfolio X by α units with cash position by 1-α units.
- **Definition 3:** A scale property of RAR is called to be scale-invariance if the RAR denoted by R(·) satisfies $R(\alpha X) = R(X)$ $\forall \alpha \in (0,1]$, $\forall X \in S$.
- **Definition 4:** A scale property of RAR is called to be scale-homogeneity if the RAR denoted by R(·) satisfies $R(\alpha X) = \alpha R(X)$ $\forall \alpha \in (0,1]$, $\forall X \in S$.
  - **Example 1:** RORC with positive-homogeneous risk measure has scale-invariance.
  - **Example 2:** EIC with positive-homogeneous risk measure has scale-homogeneity.

11. Validation Theorem for Scale-invariant RARs

- **Theorem 1:** Scale-invariant RARs have no congruent EU models.
- Proof: Let X be a bounded r.v., whose supremum of support is S_X.

  Let a scale-invariant RAR have a congruent EU model satisfying

  (1.1) $E[u(X)] = E[u(\alpha X)] > 0 \ \forall \alpha \in (0,1]$.

  A contradiction is induced by

  (1.2) $\exists \alpha^* \in (0,1]$ s.t. $E[u(\alpha^* X)] < u(\alpha^* S_X) < E[u(X)]$.

12. Validation Theorem for Scale-homogeneous RARs

- **Theorem 2:** Scale-homogeneous RARs have congruent EU models only in the case of adopting 1st LPRM as the risk measure, where the utility is linear.
- Proof: Let X be a bounded discrete r.v. with finite dimensional support V_X.
  An EU model congruent with a scale-homogeneous RAR, if any, satisfies
  (2.1) $\dim(\mathrm{Span}\{u(\alpha V_X/R(X)) - Z(\alpha);\ \forall \alpha \in (0,1]\}) = 1,\ Z(\alpha) = E[u(\alpha X/R(X))]$.
  After a brief discussion of Cauchy-type functional equation , we have
  (2.2) $Z(\alpha) = \alpha^k \left(b \int_+ X^k dP + c \int_- X^k dP \right) / R(X)^k \quad b,c > 0$.
  Since $R(\cdot)$ has mean term ($\int X\, dP$), (2.2) must satisfy k=1, b=1 and c > 1.
  Thus, we have
  $$R(X) = \int X\, dP + (c - 1) \int_- X\, dP$$ (i.e., EIC with 1st LPRM).

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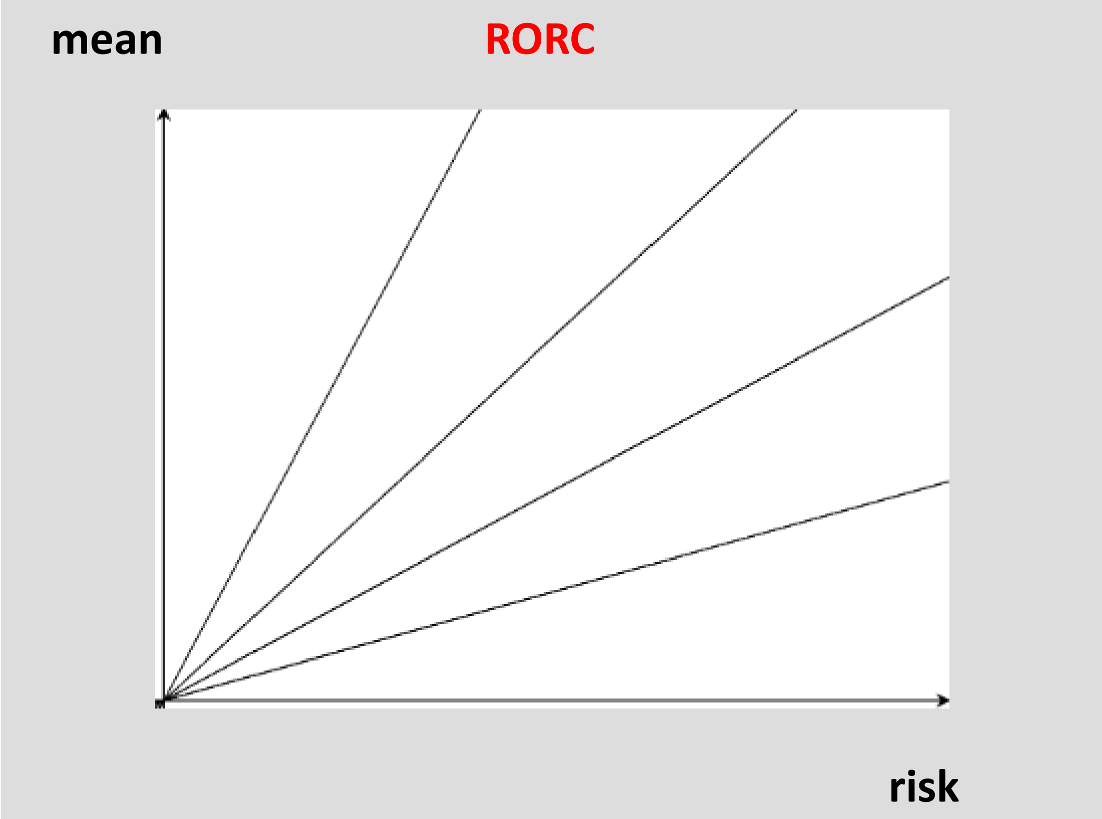
13. Search for Alternative Validation Approach

- Optimizing scale-invariant RORC is invalid while optimizing scale-homogeneous EIC is valid in the case of adopting 1st LPRM.
- Validations by scale-properties, which assume positive-homogeneity of risk measures used in RARs, are practically acceptable but may detract from generality.
- We remove the assumptions concerning scale properties and focus on RAR's indifference curves on the corresponding risk-mean plane.

14. Indifference Curves of RARs

Every RORC has a singular point at the origin where all indifference curves cross.



15. Validation Theorem by Indifference Curves

- **Theorem 3:** RORC has no congruent EU models, thus optimizing RORC is invalid.
- Proof: The congruence condition can be rewritten as

$$R(X) = R(Y) \textit{ iff } \mathbb{E}[u(X)] = \mathbb{E}[u(Y)].$$

All of the vertical intercepts of RORC's indifference curves are reduced to the origin of the risk-mean plane, while EU's indifference curves have non-zero vertical intercepts corresponding to their non-zero certainly equivalent values. Thus, RORC has no congruent EU models.

Summary of Consequences

RAR Type	Risk Measure Type	Validation Results of Optimization (Referred Portfolios)
RORC Division Type	ALL	Invalid (Low Risk)
RORC Division Type	Positive Homogeneous Type (e.g. VaR, t-VaR)	Invalid (ALL, with Cash)
EIC Subtraction Type	1st LPRM (= ∫_X dP)	Valid (ALL, with Cash)
EIC Subtraction Type	Positive Homogeneous Type, Excluding 1st LPRM	Invalid (ALL, with Cash)

Conclusion

- Above consequences create a concern that a strategic risk may be caused by optimizing RARs in spite of favorable recommendations by rating agencies.
- It can be said that we must not expect too much from RAR which is actually nothing more than a simplified indicator.
- Decision about business optimization is inherently a matter of risk appetite which has no stereotype solution.
- An ultimate goal of the risk appetite may be restated as defining the organization’s actual utility.

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"There is no universally accepted definition of ERM and the very nature of the concept suggests that there may never be one."

IAA(2009) Note on ERM for Capital and Solvency Purposes in the Insurance Industry

Thank you for your attention!

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